The GIT-equivalence for G-line bundles

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Introduction

Let G be a reductive linear algebraic group acting algebraically on a projective variety X, both defined over an algebraically closed field k. Let L be an ample G-linearized line bundle on X. Geometric Invariant Theory (GIT) associates to L a "quotient" $Y(L)$ of X by G:

$$Y(L) = \mathrm{Proj}\left(\bigoplus_{n\geq 0}\Gamma(X, L^{\otimes n})^G\right). \tag{1}$$

There is a natural G-invariant rational map $\pi : X - - \to Y(L)$. The set where π is defined is:

$$X^{\mathrm{ss}}(L) = \left\{x \in X : \exists n > 0 \text{ and } \sigma \in \Gamma(X, L^{\otimes n})^G \text{ such that } \sigma(x) \neq 0\right\}.$$

Points of $X^{\mathrm{ss}}(L)$ are said to be *semistable* for L. The map π is obtained by gluing together categorical quotients of open affine G-stable subsets of $X^{\mathrm{ss}}(L)$. As a consequence, each fiber of π contains a unique closed G-orbit in $X^{\mathrm{ss}}(L)$, and $(Y(L), \pi)$ is a categorical quotient. In particular, $Y(L)$ only depends on $X^{\mathrm{ss}}(L)$ and is denoted by $X^{\mathrm{ss}}(L)//G$. We also define the open set of *stable* points:

$$X^{\mathrm{s}}(L) = \{x \in X^{\mathrm{ss}}(L) \text{ such that } G_x \text{ is finite and } G\cdot x \text{ is closed in } X^{\mathrm{ss}}(L)\},$$

where G_x is the stabilizer of x. It turns out that $\pi^{-1}(\pi(x)) = G\cdot x$, for all $x \in X^{\mathrm{s}}(L)$. We refer to [7] or [8] for the classical properties of this quotient.

Observe that this GIT-quotient is not canonical: it depends on a choice of an ample G-linearized line bundle L over X (sometimes called a polarization of X). During the last ten years, the question of variation of quotient $X^{\mathrm{ss}}(L)//G$ under change of the ample $G-$linearized line bundle L has been

an active research subject. M. Brion and C. Procesi (see [1]) for a torus action and M. Thaddeus (see [12]) for the action of a reductive group have studied the question when the linearization varies but the line bundle does not. Recently, I. Dolgachev and Y. Hu obtained in [2] some results when L runs over all ample G-linearized line bundles. They introduced the notion of GIT-equivalence: two ample G-linearized line bundles L_1 and L_2 are said to be *GIT-equivalent* if and only if $X^{\mathrm{ss}}(L_1) = X^{\mathrm{ss}}(L_2)$. Using G-homological equivalence, they introduced a convex cone in a finite dimensional real vector space: the *G-ample cone*. The rational points of this cone give all GIT-quotients. The first theorem of [2] shows that only a finite number of GIT-quotients can be obtained when L varies. A second problem is to understand the geometry of the variation of quotient. For this, given three ample G-linearized line bundles L_-, L_0, L_+ such that $X^{\mathrm{ss}}(L_-) \subset X^{\mathrm{ss}}(L_0) \supset X^{\mathrm{ss}}(L_+)$ one studies the transformation:

$$\begin{tikzcd}
X^{\mathrm{ss}}(L_-)//G \arrow[rr, dashed] \arrow[dr, "\phi_-"'] & & X^{\mathrm{ss}}(L_+)//G \arrow[dl, "\phi_+"] \\
& X^{\mathrm{ss}}(L_0)//G &
\end{tikzcd}$$

where the morphisms $\phi_\pm$ are induced by inclusions. In [13], C. Walter gives a local model in etale topology for such a transformation.

The goal of this paper is to study the geometry of the GIT-equivalence classes and its links with the inclusions $X^{\mathrm{ss}}(L_1) \subset X^{\mathrm{ss}}(L_2)$. More precisely, if $C^G(X)$ denotes the G-ample cone, we show the following:

Theorem *Let G be a reductive algebraic group acting algebraically on a normal projective variety X. Then:*

(i) For all $l_0 \in C^G(X)$,

$$C(l_0) = \left\{ l \in C^G(X) \text{ such that } X^{\mathrm{ss}}(l_0) \subset X^{\mathrm{ss}}(l) \right\}$$

is a closed convex rational polyhedral cone in $C^G(X)$.

(ii) The cones $C(l)$ form a fan covering $C^G(X)$ (the GIT-fan).

(iii) The GIT-equivalence classes are the relative interior of these cones.

We also study the relations between the geometry of the GIT-equivalence classes and the fibers of the morphisms $X^{\mathrm{ss}}(L_1)//G \longrightarrow X^{\mathrm{ss}}(L_2)//G$ induced by inclusions $X^{\mathrm{ss}}(L_1) \subset X^{\mathrm{ss}}(L_2)$. We only use Geometric Invariant Theory contained in [8] and some classical results on instability. In particular, our arguments are valid over any algebraically closed field and for any normal variety X.

We begin with recalling the numerical criterion of stability due to Mumford. Then, we give some properties of this criterion due to Kempf, Kirwan, Mumford, Ness, Hesselink etc. The first section ends with the finiteness theorem of I. Dolgachev and Y. Hu. Following [2] and [12], in a second section we introduce the algebraic equivalence for G-linearized line bundles and the G-ample cone. The third section is devoted to the study of stability in the closure of an orbit. In the fourth and fifth sections, we study the geometry of the GIT-classes. We summarize our results by introducing the GIT-fan in Theorem 4. Finally, we fix our attention on the fibers of morphisms ϕ induced by inclusions of sets of semistable points: we describe these fibers in the case where G is a torus or $\mathrm{SL}(2)$ and we give examples for $G = k^* \times \mathrm{SL}(2)$.

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1 The numerical criterion

In this section we collect the notions and the results of Geometric Invariant Theory (see [8]) which will be used throughout this paper. We work over an algebraically closed field k of arbitrary characteristic.

1.1 The functions μ

Let G be a reductive linear algebraic group acting algebraically on a normal irreducible projective variety X. Like in [8], we denote by $\mathrm{Pic}^G(X)$ the group of G-linearized line bundles on X. Let $L \in \mathrm{Pic}^G(X)$. Let x be a point in X and λ be a one-parameter subgroup of G. Since X is complete, $\lim_{t\to 0} \lambda(t)x$ exists; let x_0 denote this limit. The image of λ fixes x_0 and so the group

$k^\times$ acts via λ on the fiber L_{x_0}. This action defines a character of $k^\times$, that is, an element of $\mathbb{Z}$ denoted by $\mu^L(x,\lambda)$. The numbers $\mu^L(x,\lambda)$ satisfy the following properties:

(i) $\mu^L(g\cdot x, g\cdot\lambda\cdot g^{-1}) = \mu^L(x,\lambda)$ for any $g \in G$;

(ii) for fixed x and λ, the map $L \mapsto \mu^L(x,\lambda)$ is an homomorphism from $\mathrm{Pic}^G(X)$ to $\mathbb{Z}$.

The numbers $\mu^L(x,\lambda)$ are used in [8] to give a numerical criterion for stability with respect to an ample $G-$linearized line bundle L:

$$\begin{array}{l} x \in X^{\mathrm{ss}}(L) \iff \mu^L(x,\lambda) \leq 0 \text{ for all one-parameter subgroups } \lambda, \\ x \in X^{\mathrm{s}}(L) \iff \mu^L(x,\lambda) < 0 \text{ for all non trivial } \lambda. \end{array}$$

1.2 Definition of the functions $\mathrm{M}^\bullet(x)$

Let T be a maximal torus of G. Denote the set of one-parameter subgroups of T (resp. G) by $\mathcal{X}_*(T)$ (resp. $\mathcal{X}_*(G)$). We denote the real vector space $\mathcal{X}_*(T)\otimes\mathbb{R}$ by $\mathcal{X}_*(T)_\mathbb{R}$. The Weyl group W of T acts linearly on $\mathcal{X}_*(T)_\mathbb{R}$. Since W is finite, there exists a W-invariant Euclidean norm $\|\cdot\|$ on $\mathcal{X}_*(T)_\mathbb{R}$. On the other hand, if $\lambda \in \mathcal{X}_*(G)$ there exists $g \in G$ such that $g\cdot\lambda\cdot g^{-1} \in \mathcal{X}_*(T)$. Moreover, if two elements of $\mathcal{X}_*(T)$ are conjugated by an element of G, then they are by an element of the normalizer of T (see [5]). This allows us to define the norm of λ by $\|\lambda\| = \|g\cdot\lambda\cdot g^{-1}\|$.

Let $L \in \mathrm{Pic}^G(X)$. We can now introduce the following notations:

$$\overline{\mu}^L(x,\lambda) = \tfrac{\mu^L(x,\lambda)}{\|\lambda\|}, \quad \mathrm{M}^L(x) = \sup_{\lambda\in\mathcal{X}_*(G)} \overline{\mu}^L(x,\lambda).$$

Actually, it is shown in [2] that $\mathrm{M}^L(x)$ is finite. The functions $\mathrm{M}^\bullet(x) : \mathrm{Pic}^G(X) \longrightarrow \mathbb{R}$ will play a central role in the rest of the paper.

1.3 $\mathrm{M}^\bullet(x)$ for a torus action

In this subsection we assume that $G = T$ is a torus. Let L be an ample T-linearized line bundle on X. Then, there exist a T-module V and a positive integer n such that $X \subset \mathbb{P}(V)$ and $L^{\otimes n} = \mathcal{O}(1)_{|X}$. Replacing L by $L^{\otimes n}$, we

assume that $n = 1$. We set $V_\chi = \{v \in V \text{ such that } \forall t \in T \quad t.v = \chi(t)v\}$. Then, we have:

$$V = \bigoplus_{\chi \in \mathcal{X}^*(T)} V_\chi.$$

Let $x \in X$ and $v \in V$ such that $[v] = x$. There exist unique vectors $v_\chi \in V_\chi$ such that $v = \sum_\chi v_\chi$. If $\lambda \in \mathcal{X}_*(T)$ then there exists, for all $\chi \in \mathcal{X}^*(T)$, an integer $< \lambda, \chi >$ such that for all $t \in T$, we have:

$$\lambda(t) \cdot v = \sum_{\chi \in \mathcal{X}^*(T)} t^{<\lambda,\chi>} v_\chi.$$

We identify $\mathcal{X}^*(T)$ and $\mathcal{X}_*(T)$ with $\mathbb{Z}^r$ in such a way that $< \cdot, \cdot >$ is the standard inner product in $\mathbb{Z}^r$.

Set $\mathrm{st}(x) = \{\chi \in \mathcal{X}^*(T) \text{ such that } v_\chi \neq 0\}$. We have:

$$\mu^L(x, \lambda) = \min_{\chi \in \mathrm{st}(x)} < \lambda, \chi > .$$

The following proposition due to L.Ness gives a pleasant interpretation of the number $\mathrm{M}^L(x)$:

Proposition 1 *(see [9]) With the above notations, we have:*

- *(i) The point x is unstable if and only if 0 does not belong to the convex hull of $\mathrm{st}(x)$. In this case, $\mathrm{M}^L(x)$ is the distance from 0 to this convex hull.*
- *(ii) If x is semistable, the opposite of $\mathrm{M}^L(x)$ is the distance from 0 to the boundary of this convex set.*
- *(iii) There exists $\lambda \in \mathcal{X}_*(T)$ such that $\overline{\mu}^L(x, \lambda) = \mathrm{M}^L(x)$. If moreover λ is indivisible, we call it an adapted one-parameter subgroup for x.*
- *(iv) If x is unstable, there exists a unique adapted one-parameter subgroup for x.*

1.4 Properties of $M^\bullet(x)$

Lemma 1 *(see [9]) Let L be an ample G-linearized line bundle and T be a maximal torus of G. We denote by $r_T : \mathrm{Pic}^G(X) \to \mathrm{Pic}^T(X)$ the partial forgetful map.*

Then, for all $x \in X$, the set of the numbers $\mathrm{M}^{r_T(L)}(g \cdot x)$ for $g \in G$ is finite and $\mathrm{M}^L(x) = \max_{g \in G} \mathrm{M}^{r_T(L)}(g \cdot x)$.

An indivisible one-parameter subgroup λ of G is said to be *adapted for x and L* if and only if $\overline{\mu}^L(x,\lambda) = \mathrm{M}^L(x)$. Denote by $\Lambda^L(x)$ the set of adapted one-parameter subgroups for x.

Corollary 1 *(i) The numbers $M^L(x)$ are finite (even if L is not ample, see Proposition 1.1.6 in [2]).*

(ii) If L is ample, $\Lambda^L(x)$ is not empty.

Now, we can reformulate the numerical criterion for stability: if L is ample, we have

$$X^{\mathrm{ss}}(L) = \{x \in X : \mathrm{M}^L(x) \leq 0\}, \quad X^{\mathrm{s}}(L) = \{x \in X : \mathrm{M}^L(x) < 0\}.$$

To a one-parameter subgroup λ of G we associate the parabolic subgroup (see [8]):

$$P(\lambda) = \left\{ g \in G \text{ such that } \lim_{t \to 0} \lambda(t).g.\lambda(t)^{-1} \text{ exists in } G \right\}.$$

Then, for $g \in P(\lambda)$, we have $\mu^L(x,\lambda) = \mu^L(x, g.\lambda.g^{-1})$. The following theorem due to G. Kempf is a generalization of the last assertion of Proposition 1.

Theorem 1 *(see [6]) Let x be an unstable point for an ample G-linearized line bundle L. Then:*

(i) All the $P(\lambda)$ for $\lambda \in \Lambda^L(x)$ are equal. We denote by $P^L(x)$ this subgroup.

(ii) Any two elements of $\Lambda^L(x)$ are conjugated by an element of $P^L(x)$.

We will also use the following theorem of L. Ness.

Theorem 2 *(see [10]) Let x and L be as in the above theorem. Let λ be an adapted one-parameter subgroup for x and L. We consider $y = \lim_{t\to 0} \lambda(t) \cdot x$. Then:*

(i) $\lambda \in \Lambda^L(y)$,

(ii) $\mathrm{M}^L(x) = \mathrm{M}^L(y)$.

1.5 Stratification of X induced from L

If $d > 0$ and $< \tau >$ is a conjugacy class of one-parameter subgroups of G, we set:

$$S^L_{d,<\tau>} = \left\{x \in X \text{ such that } \mathrm{M}^L(x) = d \text{ and } \Lambda^L(x) \cap < \tau > \neq \emptyset\right\}.$$

If $\mathcal{T}$ is the set of conjugacy classes of one-parameter subgroups, the previous section gives us the following decomposition of X:

$$X = X^{\mathrm{ss}}(L) \cup \bigcup_{d>0,\ <\tau>\in\mathcal{T}} S^L_{d,<\tau>}.$$

W. Hesselink showed in [4] that this union is a finite stratification by G-stable locally closed subvarieties of X. We will call it the *stratification induced from L.*

Using this stratification, I. Dolgachev and Y. Hu have shown the following fundamental finiteness theorem (see Theorem 1.3.9 in [2]):

Theorem 3 *(i) The set of locally closed subvarieties S of X which can be realized as the stratum $S^L_{d,<\tau>}$ for some ample $L \in \mathrm{Pic}^G(X), d > 0$ and $\tau \in \mathcal{X}_*(G)$ is finite.*

(ii) The set of possible open subsets of X which can be realized as the set of semistable points with respect to some ample G-linearized line bundle is finite.

2 The G-ample cone

2.1 Algebraic equivalence for G-line bundles

Following I. Dolgachev and Y. Hu, we introduce the G-ample cone. As M. Thaddeus in [12], we use algebraic equivalence of G-line bundles instead of homological equivalence.

Two elements L_1 and L_2 of $\mathrm{Pic}^G(X)$ are said to be *G-algebraically equivalent* if there exist a connected variety S, points t_1, $t_2 \in S$, and a G-linearized line bundle L on $S \times X$ such that $L_{|\{t_1\}\times X} = L_1$ and $L_{|\{t_2\}\times X} = L_2$. Here, G acts on $S \times X$ via its action on the second factor. Let $\mathrm{NS}^G(X)$ denote the quotient of $\mathrm{Pic}^G(X)$ by this equivalence relation. The following proposition is analogous to Lemma 2.3.5 in [2].

Proposition 2 *Let $x \in X$ and $\lambda \in \mathcal{X}_*(G)$. Let L_1, $L_2 \in \mathrm{Pic}^G(X)$ be G-algebraically equivalent. Then, $\mu^{L_1}(x,\lambda) = \mu^{L_2}(x,\lambda)$.*

Proof : Let S, L, t_1 and t_2 be as in the definition of the G-algebraic equivalence. Denote by y the point $\lim_{t\to 0}\lambda(t)\cdot x$. The group $k^\times$ acts via λ on the fibers $L_{(t,y)}$ for all $t \in S$. Consider the map $S \longrightarrow \mathbb{Z}$, $t \mapsto \mu^L((t,y),\lambda)$. Obviously, this map is locally constant. By connectness of S we obtain $\mu^{L_1}(x,\lambda) = \mu^L((t_1,y),\lambda) = \mu^L((t_2,y),\lambda) = \mu^{L_2}(x,\lambda)$. □

We denote by $\mathrm{NS}^G(X)_\mathbb{R}$ (resp. $\mathrm{NS}^G(X)_\mathbb{Q}$) the vector space $\mathrm{NS}^G(X)\otimes\mathbb{R}$ (resp. $\mathrm{NS}^G(X)\otimes\mathbb{Q}$). A point $l \in \mathrm{NS}^G(X)_\mathbb{R}$ is said to be *rational* if it belongs to $\mathrm{NS}^G(X)_\mathbb{Q}$. The forgetful map $\mathrm{Pic}^G(X) \longrightarrow \mathrm{Pic}(X)$ descends to $f : \mathrm{NS}^G(X) \longrightarrow \mathrm{NS}(X)$. We denote by $\mathcal{X}^*(G)_\mathbb{R}$ (resp. $\mathcal{X}^*(G)_\mathbb{Q}$) the vector space $\mathcal{X}^*(G)\otimes\mathbb{R}$ (resp. $\mathcal{X}^*(G)\otimes\mathbb{Q}$). M. Thaddeus has proved in [12] the following:

Proposition 3 *The map f induces an exact sequence:*

$$0 \longrightarrow \mathcal{X}^*(G)_\mathbb{R} \longrightarrow \mathrm{NS}^G(X)_\mathbb{R} \longrightarrow \mathrm{NS}(X)_\mathbb{R} \longrightarrow 0$$

In particular, $\mathrm{NS}^G(X)_\mathbb{R}$ is a finite dimensional real vector space. Proposition 2 allows us to define $\mu^\bullet(x,\lambda) : \mathrm{NS}^G(X) \longrightarrow \mathbb{Z}$ and $\mathrm{M}^\bullet(x) : \mathrm{NS}^G(X) \longrightarrow \mathbb{R}$. By linearity, we can define $\mu^\bullet(x,\lambda) : \mathrm{NS}^G(X)_\mathbb{R} \longrightarrow \mathbb{R}$. The function $\mathrm{M}^\bullet(x) : \mathrm{NS}^G(X)_\mathbb{R} \longrightarrow \mathbb{R}$ is defined by the formula $\mathrm{M}^l(x) = \sup_\lambda \overline{\mu}^l(x,\lambda)$.

Lemma 2 *Let $x \in X$. The function $\mathrm{NS}^G(X)_\mathbb{R} \longrightarrow \mathbb{R}$, $l \mapsto \mathrm{M}^l(x)$ is convex and positively homogeneous. In particular, it is continuous.*

Proof : The functions $\mathrm{M}^\bullet(x)$ are the suprema of a family of linear forms. □

If $l \in \mathrm{NS}^G(X)_\mathbb{R}$, we set

$$X^{\mathrm{ss}}(l) = \{x \in X \text{ such that } \mathrm{M}^l(x) \leq 0\},$$
$$X^{\mathrm{s}}(l) = \{x \in X \text{ such that } \mathrm{M}^l(x) < 0\}.$$

We denote by $\mathrm{NS}^G(X)^+_\mathbb{R}$ the convex cone generated by the classes of ample G-linearized line bundles in $\mathrm{NS}^G(X)_\mathbb{R}$. This cone is open in $\mathrm{NS}^G(X)_\mathbb{R}$. Indeed, it is the preimage by f of the ample cone, a strictly convex open

cone in $\mathrm{NS}(X)_{\mathbb{R}}$. A point l in $\mathrm{NS}^G(X)_{\mathbb{R}}$ is said to be *ample* if and only if it belongs to $\mathrm{NS}^G(X)^+_{\mathbb{R}}$.

A point $l \in \mathrm{NS}^G(X)^+_{\mathbb{R}}$ is said to be *effective* if and only if $X^{\mathrm{ss}}(l)$ is not empty. The set of effective points of $\mathrm{NS}^G(X)^+_{\mathbb{R}}$ is denoted by $C^G(X)$ and called the *G-ample cone*. It is shown in [2] that the G-ample cone is convex. It may happen that $C^G(X)$ does not generate $\mathrm{NS}^G(X)_{\mathbb{R}}$; for example if G is a product $H \times k^*$ and k^* acts trivially on X.

Two points l and l' in $C^G(X)$ are said to be *GIT-equivalent* if and only if $X^{\mathrm{ss}}(l) = X^{\mathrm{ss}}(l')$. If $l_0 \in \mathrm{NS}^G(X)^+_{\mathbb{R}}$, the *GIT-class* of l_0 is the set of all $l \in \mathrm{NS}^G(X)^+_{\mathbb{R}}$ that are GIT-equivalent to l_0. The purpose of this paper is to describe the map $l \in C^G(X) \mapsto X^{\mathrm{ss}}(l)$. More precisely, we will describe the geometry of the GIT-classes.

Remark : Let $l \in C^G(X)$. If $l \in \mathrm{NS}^G(X)_{\mathbb{Q}}$, there exists a G-linearized line bundle L on X such that $X^{\mathrm{ss}}(l) = X^{\mathrm{ss}}(L)$ and $X^{\mathrm{s}}(l) = X^{\mathrm{s}}(L)$. So, we have an algebraic quotient $X^{\mathrm{ss}}(l)//G$ with all the properties of Geometric Invariant Theory. For example, if l is rational then the sets $X^{\mathrm{ss}}(l)$ and $X^{\mathrm{s}}(l)$ are open and a point $x \in X^{\mathrm{ss}}(l)$ belongs to $X^{\mathrm{s}}(l)$ if and only if its stabilizer is finite and its orbit is closed in $X^{\mathrm{ss}}(l)$. At this step of the paper, we do not know if these properties hold for $X^{\mathrm{ss}}(l)$ and $X^{\mathrm{s}}(l)$ when l is not rational in $C^G(X)$. This will often lead us to assume that: l is rational. Actually, this assumption will turn out to be unnecessary because Proposition 7 will show that any l in $C^G(X)$ is GIT-equivalent to a rational one.

2.2 A first property of the map $l \mapsto X^{\mathrm{ss}}(l)$

The following proposition is a result of local monotonicity of the maps $l \mapsto X^{\mathrm{ss}}(l)$ and $l \mapsto X^{\mathrm{s}}(l)$.

Proposition 4 *Let l_0 be a point in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$. There exists a neighborhood V of l_0 such that $X^{\mathrm{s}}(l_0) \subset X^{\mathrm{s}}(l) \subset X^{\mathrm{ss}}(l) \subset X^{\mathrm{ss}}(l_0)$ for all rational $l \in V$.*

Proof : By Theorem 3, there exist finitely many open subsets $X^{\mathrm{s}}_1, \dots, X^{\mathrm{s}}_n$ of X such that $X^{\mathrm{s}}(l)$ is one of them for all $l \in \mathrm{NS}^G(X)^+_{\mathbb{Q}}$. We order these sets such that:

$$\begin{array}{rl} \text{(i)} & X^{\mathrm{s}}(l_0) \not\subset X^{\mathrm{s}}_i \text{ for } i = 1, \dots, p \\ \text{(ii)} & X^{\mathrm{s}}(l_0) \subset X^{\mathrm{s}}_i \text{ for } i = p+1, \dots, n. \end{array}$$

Let us fix some points $x_1, \dots, x_p$ in $X^{\mathrm{s}}(l_0)$ such that $x_i \notin X_i^{\mathrm{s}}$ for all $i = 1, \dots, p$. By continuity of the functions $\mathrm{M}^\bullet(x)$, there exist neighborhoods $V_{x_1}, \dots, V_{x_p}$ of l_0 in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$ such that:

$$\forall l \in V_{x_i} \cap \mathrm{NS}^G(X)_{\mathbb{Q}} \quad x_i \in X^{\mathrm{s}}(l).$$

Let V denote the intersection of the V_{x_i}. For all $l \in V \cap \mathrm{NS}^G(X)_{\mathbb{Q}}$, $X^{\mathrm{s}}(l)$ is different from X_i^{s} for all $i = 1, \dots, p$. Therefore, $X^{\mathrm{s}}(l)$ is one of the sets $X^{\mathrm{s}}_{p+1}, \dots, X^{\mathrm{s}}_n$ and contains $X^{\mathrm{s}}(l_0)$.

Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of balls centered at l_0 and contained in V such that the sequence of radii converges to zero. Let $X_1^{\mathrm{ss}}, \dots, X_k^{\mathrm{ss}}$ be the open subsets of X such that $X^{\mathrm{ss}}(l)$ is one of them for all $l \in \mathrm{NS}^G(X)^+_{\mathbb{Q}}$. We set:

$$I_n = \left\{ i \mid 1 \leq i \leq k, \exists l \in B_n \cap \mathrm{NS}^G(X)_{\mathbb{Q}} \text{ such that } l \neq l_0 \text{ and } X^{\mathrm{ss}}(l) = X_i^{\mathrm{ss}} \right\}$$

The sequence (I_n) is decreasing. Since the sets I_n are finite, the sequence I_n is stationary (from a rank N).

Let $l \in B_N \cap \mathrm{NS}^G(X)_{\mathbb{Q}}$ and $x \in X^{\mathrm{ss}}(l)$. For all n, there exists $l_n \in B_n$ such that $x \in X^{\mathrm{ss}}(l_n) = X^{\mathrm{ss}}(l)$. Since the sequence l_n converges to l_0 and the function $\mathrm{M}^\bullet(x)$ is continuous, we have $\mathrm{M}^{l_0}(x) \leq 0$; that is $x \in X^{\mathrm{ss}}(l_0)$. And so $X^{\mathrm{ss}}(l) \subset X^{\mathrm{ss}}(l_0)$.

$\square$

3 The stability set of a point

Definition If $x \in X$, the set $\left\{ l \in \mathrm{NS}^G(X)^+_{\mathbb{R}} \text{ such that } x \in X^{\mathrm{ss}}(l) \right\}$ is denoted by $\Omega(x)$. Observe that $\Omega(x) = \{ l \in \mathrm{NS}^G(X)^+_{\mathbb{R}} \text{ such that } \mathrm{M}^l(x) \leq 0 \}$. The set $\Omega(x)$ is called *the stability set of* x.

The aim of this section is to describe the geometry of the stability sets. In other words, we fix a point $x \in X$ and study the semistability of this point when l varies in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$. Let $\overline{G \cdot x}$ denote the closure of the orbit of x. In fact, the study of $\Omega(x)$ will lead us to consider the GIT-classes of the G-variety $\overline{G \cdot x}$.

3.1 A lemma on μ

Lemma 3 *Let l be a rational point in the G-ample cone. Let $x \in X^{ss}(l)$ and $\lambda \in \mathcal{X}_*(G)$. We denote by z the point $\lim_{t\to 0} \lambda(t)x$.*

If z is unstable for l then $\mu^l(x, \lambda) < 0$.

Proof : There exist a G-module V and a positive integer n such that X can be embedded into $\mathbb{P}(V)$ and the algebraic equivalence class of the restriction of $\mathcal{O}(1)$ to X is $l^{\otimes n}$. Replacing l by $l^{\otimes n}$, we may assume that $n = 1$.

Let us assume that $\mu^l(x,\lambda)$ is non negative. Since x is semistable for l, we have $\mu^l(x,\lambda) = 0$. Let T be a maximal torus in G such that $\mathrm{Im}(\lambda) \subset T$, and let $g \in G$. We consider $\lambda' = g\lambda g^{-1}$, $x' = g \cdot x$ and $z' = g \cdot z$. Then, $\mu^l(x',\lambda') = 0$. We use the notations of Section 1.3. Since $z' = \lim_{t\to 0} \lambda'(t)x'$, $\mathrm{st}(z')$ is equal to the intersection of $\mathrm{st}(x')$ and $\{\chi$ such that $<\chi,\lambda'>= \mu^l(x',\lambda')\}$. So the set $\mathrm{st}(z')$ is a face of $\mathrm{st}(x')$ containing 0. In particular, z' is semistable for $r_T(l)$. Finally, z is semistable for l. This proves the lemma. $\square$

3.2 A first result of rationality

Proposition 5 *Let $x \in X$. The stability set $\Omega(x)$ of x is a convex cone and is closed in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$. Moreover, the span of $\Omega(x)$ is a rational vector subspace of $\mathrm{NS}^G(X)_{\mathbb{R}}$. In particular, $\Omega(x)$ is the closure of its rational points.*

Proof : The first assertion is obvious because the function $\mathrm{M}^\bullet(x)$ is convex and positively homogeneous. The last assertion is a direct consequence of the first ones. Let us prove the second one. Let $F \subset \mathrm{NS}^G(X)_{\mathbb{R}}$ be the minimal rational vector subspace such that $\Omega(x)$ is contained in F. Suppose that $\Omega(x)$ does not span F.

Since $\Omega(x)$ is convex, its interior as a subset of F is empty. This implies that $\mathrm{M}^l(x) = 0$, for all $l \in \Omega(x)$. Let l be a point in $\Omega(x)$. There exists a sequence (l_n) of points in F not in $\Omega(x)$ which converges to l. Since F is rational, we may assume that all l_n are rational points.

By Theorem 3, by extracting a subsequence, we may assume that all l_n induce the same stratification s. For all n, x is unstable for l_n. So there exists a non open stratum S of s containing x. Let $\lambda_0 \in \Lambda^{l_0}(x)$ and $y = \lim_{t\to 0} \lambda_0(t) \cdot x$. By Theorem 2, $y \in S$, and so for all n we have $\mathrm{M}^{l_n}(x) = \mathrm{M}^{l_n}(y)$. But now, the continuity of the functions $\mathrm{M}^\bullet(x)$ and $\mathrm{M}^\bullet(y)$ implies that $\mathrm{M}^l(y) = \mathrm{M}^l(x) = 0$.

Since λ_0 fixes y, we have $\mu^l(y,-\lambda_0) = -\mu^l(y,\lambda_0)$. So, $\mathrm{M}^l(y) = 0$ implies $\mu^l(x,\lambda_0) = \mu^l(y,\lambda_0) = 0$. But $\mu^{l_0}(x,\lambda_0) > 0$, so $\mu^\bullet(x,\lambda_0)$ is not zero on F. Therefore, the equation $\mu^\bullet(x,\lambda_0) = 0$ defines an hyperplane of F containing l.

Moreover, the functions $\mu^\bullet(x,\lambda)$ are rational on $\mathrm{NS}^G(X)_\mathbb{Q}$. In particular, the set of the functions equal to $\mu^\bullet(x,\lambda)$ for some one-parameter subgroup λ is countable. Hence, $\Omega(x)$ is contained in a countable union of hyperplanes of F. But $\Omega(x)$ is convex, thus $\Omega(x)$ is contained in such an hyperplane. Therefore, $\Omega(x)$ is contained in a rational hyperplane of F. Since F is minimal, this is a contradiction. The second assertion of the proposition is proved. $\square$

Note that Proposition 5 is proved in [2] in the special case when the codimension of $\Omega(x)$ is equal to one.

Corollary 2 *The number of stability sets is finite.*

Proof : It is clear that the stability set of each point of X is an union of GIT-classes. So, by Theorem 3, there exists only a finite number of sets of the form $\Omega(x)\cap\mathrm{NS}^G(X)_\mathbb{Q}$ for $x\in X$. Since Proposition 5 shows that $\Omega(x)$ is the closure of $\Omega(x)\cap\mathrm{NS}^G(X)_\mathbb{Q}$, the corollary is proved. $\square$

We also mention the well-known (see [2])

Corollary 3 *The G-ample cone $C^G(X)$ is closed in $\mathrm{NS}^G(X)^+_\mathbb{R}$.*

Proof : Since $C^G(X)=\cup_{x\in X}\Omega(x)$, Proposition 5 and Corollary 2 imply that $C^G(X)$ is closed in $\mathrm{NS}^G(X)^+_\mathbb{R}$. $\square$

3.3 Geometry of $\Omega(x)$

The following lemma is essential in the study of the geometry of the stability sets.

Lemma 4 *Let $x\in X$ and $z\in\overline{G\cdot x}-G\cdot x$. We assume that there exists a rational point $l_0\in C^G(x)$ such that $G\cdot z$ is closed in $X^{ss}(l_0)$. Then, there exists $\lambda\in\mathcal{X}_*(G)$ such that*

$$
\begin{array}{rl}
(i) & \lim_{t\to 0}\lambda(t)\cdot x\in G\cdot z\\
(ii) & \Omega(x)\subset\{l\in\mathrm{NS}^G(X)_\mathbb{R}\text{ such that }\mu^l(x,\lambda)\leq 0\}\\
(iii) & \Omega(z)=\{l\in\mathrm{NS}^G(X)_\mathbb{R}\text{ such that }\mu^l(x,\lambda)=0\}\cap\Omega(x)
\end{array}
$$

Proof : The Hilbert-Mumford theorem (see [8]) applied to $X^{ss}(l_0)$ gives us a $\lambda\in\mathcal{X}_*(G)$ such that $\lim_{t\to 0}\lambda(t)\cdot x\in G\cdot z$. Denote by z' this limit. Since the image of λ fixes z', if $z'\in X^{\mathrm{ss}}(l)$ then $\mu^l(z',\lambda)$ and $\mu^l(z',-\lambda)=-\mu^l(z',\lambda)$

are negative or zero. So we have $\Omega(z) = \Omega(z') \subset \{l$ such that $\mu^l(z', \lambda) = \mu^l(x, \lambda) = 0\}$.

Let l be a rational point in the stability set of z. Since $X^{\mathrm{ss}}(l)$ is open and $z \in \overline{G \cdot x}$, we have $l \in \Omega(x)$. But now, Proposition 5 implies that $\Omega(z)$ is contained in $\Omega(x)$. The inclusion $\Omega(x) \subset \{l \in \mathrm{NS}^G(X)_{\mathbb{R}}$ such that $\mu^l(x, \lambda) \leq 0\}$ is obvious; it implies that $\Omega(z) \subset \{l \in \mathrm{NS}^G(X)_{\mathbb{R}}$ such that $\mu^l(x, \lambda) \leq 0\} \cap \Omega(x)$. We show the opposite inclusion.

Let l be a rational point in $\Omega(x)$ such that $\mu^l(x, \lambda) = 0$. Then, Lemma 3 shows that z is semistable for l. So l belongs to the stability set of z. We conclude by density of rational points in $\{l$ such that $\mu^l(x, \lambda) = 0\} \cap \Omega(x)$ (which is the intersection of $\Omega(x)$ and a rational hyperplane). $\square$

Definition A *polyhedral cone* in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$ is a subset of $\mathrm{NS}^G(X)^+_{\mathbb{R}}$ defined by a finite number of linear inequalities. Such a cone is said to be *rational* if the inequalities can be chosen to be rational. Let C be a polyhedral cone in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$. If f is a linear form on $\mathrm{NS}^G(X)_{\mathbb{R}}$ non negative on C, the set of the points c in C such that $f(c) = 0$ is said to be a *face* of C.

Proposition 6 *Let $x \in X$.*

(i) There exists $y \in \overline{G \cdot x}$ such that: l belongs to the relative interior of $\Omega(y)$ if and only if $G \cdot y$ is closed in $X^{\mathrm{ss}}(l)$. Moreover, $\Omega(x) = \Omega(y)$.

(ii) The stability set of x is a convex rational polyhedral cone in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$.

(iii) The faces of $\Omega(x)$ are exactly the sets $\Omega(y)$ with $y \in \overline{G \cdot x}$.

Proof : Let l_0 be a rational point in the relative interior of $\Omega(x)$. Let $y \in \overline{G \cdot x}$ such that $G \cdot y$ is closed in $X^{\mathrm{ss}}(l_0)$. Lemma 4 shows that $\Omega(y)$ is a face of $\Omega(x)$. Since $l_0 \in \Omega(y)$, we have $\Omega(x) = \Omega(y)$. In particular, for all $z \in X$ there exists $z' \in \overline{G \cdot z}$ such that $\Omega(z) = \Omega(z')$ and $G \cdot z'$ is closed in $X^{\mathrm{ss}}(l)$ for some rational point $l \in C^G(X)$. Now, Lemma 4 shows that the sets $\Omega(z')$ with $z' \in \overline{G \cdot x}$ are faces of $\Omega(x)$.

Let l be a point in the relative boundary of $\Omega(y)$. By Proposition 5, there exists a sequence $(l_n)_{n \geq 1}$ of rational points in the vector space spanned by $\Omega(y)$, but out of $\Omega(y)$, which converges to l. By Theorem 3, by extracting a subsequence we may assume that all l_n induce the same stratification. Now, like in the proof of Proposition 5 we choose $\lambda_l \in \Lambda^{l_1}(y)$ and we set $z_l = \lim_{t \to 0} \lambda(t) \cdot y$. Then, $z_l \in \overline{G \cdot y}$ and $l \in \Omega(z_l)$. Moreover, $\Omega(z_l)$ is

contained in the hyperplane of $\mathrm{NS}^G(X)_{\mathbb{R}}$ with equation $\mu^\bullet(y, \lambda_l) = 0$, whereas $\Omega(y)$ is not. Thus l belongs to $\Omega(z_l)$ which is a proper face of $\Omega(y)$.

We just proved that the relative boundary of $\Omega(y)$ is the union of its faces $\Omega(z)$ for some $z \in \overline{G \cdot y}$. Now, Proposition 5 and Corollary 2 imply the second assertion of the proposition.

Moreover, any face of codimension one of $\Omega(y)$ is equal to $\Omega(z)$ for some $z \in \overline{G \cdot y}$. By induction on the codimension of the face, this proves that any face of $\Omega(y)$ is equal to $\Omega(z)$ for some $z \in \overline{G \cdot y}$.

Let us prove that y satisfies the first assertion of the proposition. The above discussion shows that if l belongs to the relative boundary of $\Omega(y)$ then $G \cdot y$ is not closed in $X^{\mathrm{ss}}(l)$ (because $z_l \in (\overline{G \cdot y} - G \cdot y) \cap X^{\mathrm{ss}}(l)$). Conversely, let l' be a point in $\Omega(y)$ such that $G \cdot y$ is not closed in $X^{\mathrm{ss}}(l')$. There exists $z' \in (\overline{G \cdot y} - G \cdot y) \cap X^{\mathrm{ss}}(l')$. Since $G \cdot y$ is closed in $X^{\mathrm{ss}}(l_0)$, $z' \notin X^{\mathrm{ss}}(l_0)$ and $\Omega(z')$ is a proper face of $\Omega(y)$. Moreover, $\Omega(z')$ contains l'. Thus l' does not belong to the relative interior of $\Omega(y)$. $\square$

Definition A point x is said to be *pivotal for* l if x is semistable for l (or for the GIT-class, F of l) and $G \cdot x$ is closed in $X^{\mathrm{ss}}(l)$. A point x is said to be *pivotal for* Ω if Ω is the stability set of x and x is pivotal for some (or any) point in the relative interior of Ω.

Let us remark that I. Dolgachev and Y. Hu use in [2] a notion of pivotal point which is close to ours.

4 Geometry of the GIT-classes

4.1 A fundamental lemma

Up to now, we have studied the stability sets $\Omega(x)$ and for this the quasi-homogeneous varieties $\overline{G \cdot x}$. For an arbitrary X, the GIT-class of a point l in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$ depends on the relative positions of l and the stability sets of the points of X. To show how the various stability sets are related, the main tool is the following:

Lemma 5 *Let l_1 and l_2 be two points in the G-ample cone such that the set $X^{\mathrm{ss}}(l_1)$ is contained in $X^{\mathrm{ss}}(l_2)$. Let x be a pivotal point for l_2.*

Then there exists a pivotal point y for l_1 such that $x \in \overline{G \cdot y}$. Moreover, $X^{\mathrm{s}}(l_2)$ is contained in $X^{\mathrm{s}}(l_1)$.

Proof : Let us consider the following commutative diagram:

$$\begin{tikzcd} X^{\mathrm{ss}}(l_1) \arrow[r, hook, "\text{inclusion}"] \arrow[d, "\pi_{l_1}"] & X^{\mathrm{ss}}(l_2) \arrow[d, "\pi_{l_2}"] \\ X^{\mathrm{ss}}(l_1)//G \arrow[r, "\phi"] & X^{\mathrm{ss}}(l_2)//G \end{tikzcd}$$

The image of ϕ is equal to $\pi_{l_2}(X^{\mathrm{ss}}(l_1))$. Moreover, π_{l_2} is surjective and $X^{\mathrm{ss}}(l_1)$ is dense in $X^{\mathrm{ss}}(l_2)$. So ϕ is dominant. Since $X^{\mathrm{ss}}(l_1)//G$ is complete, ϕ is surjective.

Therefore, there exists $y \in X^{\mathrm{ss}}(l_1)$ such that $\phi(\pi_{l_1}(y)) = \pi_{l_2}(x)$. Since $y \in \pi_{l_2}^{-1}(\pi_{l_2}(x))$ and $G \cdot x$ is the unique closed orbit in $\pi_{l_2}^{-1}(\pi_{l_2}(x))$, we have $x \in \overline{G \cdot y}$. Moreover, $x \in \overline{G \cdot y'}$ for all $y' \in X^{\mathrm{ss}}(l_1) \cap \overline{G \cdot y}$. Therefore we can find y such that $G \cdot y$ is closed in $X^{\mathrm{ss}}(l_1)$ and $x \in \overline{G \cdot y}$. This proves the first assertion of the lemma.

Let x be a stable point for l_2 and let y be as above. Since G_x is finite, $G \cdot x = G \cdot y$ and x is semistable for l_1. Moreover, $G \cdot x$ is closed in $X^{\mathrm{ss}}(l_2)$ and hence in $X^{\mathrm{ss}}(l_1)$. Thus, x is stable for l_1. $\square$

4.2 The geometry of a GIT-class

Let F be a GIT-class. We denote by $X^{\mathrm{ss}}(F)$ (resp. $X^{\mathrm{s}}(F)$) the subset $X^{\mathrm{ss}}(l)$ (resp. $X^{\mathrm{s}}(l)$) of X for some (or any) $l \in F$.

Lemma 6 *Let F be a GIT-class. Then,*

$$F = \bigcap_{x \text{ pivotal for F}} \mathrm{RelInt}(\Omega(\mathrm{x}))$$

where $\mathrm{RelInt}(\Omega(x))$ *is the relative interior of the stability set of x.*

Proof : The first assertion of Proposition 6 shows that F is contained in the intersection in the lemma. Conversely, let l be a point in this intersection. Then, $X^{\mathrm{ss}}(l)$ contains each closed orbit of $X^{\mathrm{ss}}(F)$, so $X^{\mathrm{ss}}(F)$ is contained in $X^{\mathrm{ss}}(l)$. Let $x \in X^{\mathrm{ss}}(l)$ such that $G \cdot x$ is closed in $X^{\mathrm{ss}}(l)$. By Lemma 5, there exists $y \in X^{\mathrm{ss}}(F)$ such that $x \in \overline{G \cdot y}$ and $G \cdot y$ is closed in $X^{\mathrm{ss}}(F)$. By assumption, l belongs to the relative interior of the stability set of y. Then, Proposition 6 implies that $G \cdot y$ is closed in $X^{\mathrm{ss}}(l)$. So $x \in G \cdot y$, and $X^{\mathrm{ss}}(l) = X^{\mathrm{ss}}(F)$. The lemma is proved. $\square$

Proposition 7 *Any GIT-class is the relative interior of a rational polyhedral cone in* $\mathrm{NS}^G(X)^+_{\mathbb{R}}$.

Proof : The proposition follows immediately from Lemma 6, Proposition 6 and Corollary 2. $\square$

Remark

(i) This proposition implies that any point in $C^G(X)$ is GIT-equivalent to a rational point in $C^G(X)$. In particular, the second assertion of Theorem 3, Proposition 4 and Lemmas 3 and 5 hold for any real point in $C^G(X)$.

(ii) Let us assume for a moment that X is a smooth complex variety. Let K be a maximal compact subgroup of G. Let ω be a K-invariant Kählerian symplectic form on X and let ϕ be a moment map for the action of K. In this situation, $X^{\mathrm{ss}}(\omega,\phi) = \{x \in X \text{ such that } \overline{G\cdot x} \cap \phi^{-1}(0) \neq \emptyset\}$ is called the set of semistable points. Then, $X^{\mathrm{ss}}(\omega,\phi)//G$ is a complex space, homeomorphic to $\phi^{-1}(0)/K$. It turns out that $X^{\mathrm{ss}}(\omega,\phi)//G$ only depends on the class of ω in $H^2(X,\mathbb{R})$ and on the choice of ϕ (see, for example Theorem 2.3.8 of [2]). On the other hand, $\mathrm{NS}(X)_{\mathbb{R}}$ is a subspace of $H^2(X,\mathbb{R})$. The theory of Kempf-Ness shows that if L and ω have the same class in $H^2(X,\mathbb{R})$, then the complex spaces $X^{\mathrm{ss}}(\omega,\phi)//G$ and $X^{\mathrm{ss}}(L)//G$ are isomorphic. So, the previous proposition shows that if the class of ω belongs to $\mathrm{NS}(X)_{\mathbb{R}}$, the complex space $X^{\mathrm{ss}}(\omega,\phi)//G$ is a projective algebraic variety. More generally, P. Heinzner and L. Migliorini showed in [3] that for all (ω,ϕ) the set $X^{\mathrm{ss}}(\omega,\phi)$ is equal to $X^{\mathrm{ss}}(L)$ for some ample G-linearized line bundle L.

4.3 The inclusion relations $X^{\mathrm{ss}}(F) \subset X^{\mathrm{ss}}(F')$

Proposition 8 *Let F and F' be two GIT-classes. The following assertions are equivalent:*

(i) F' intersects the closure of F in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$

(ii) F' is contained in the closure of F in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$

(iii) $X^{\mathrm{ss}}(F)$ is contained in $X^{\mathrm{ss}}(F')$

Proof : It is sufficient to prove that

$$l \in \overline{F} \iff X^{\rm ss}(F) \subset X^{\rm ss}(l),$$

where $\overline{F}$ denotes the closure of F in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$.

If $l \in \overline{F}$, the continuity of the functions $\mathrm{M}^\bullet(x)$ implies that $X^{\rm ss}(F)$ is contained in $X^{\rm ss}(l)$. Conversely, let $l \in \mathrm{NS}^G(X)^+_{\mathbb{R}}$ be such that $X^{\rm ss}(F) \subset X^{\rm ss}(l)$. Let l_0 be in F. Let x be a pivotal point for F. By Lemma 6, l_0 belongs to the relative interior of $\Omega(x)$. Moreover, the closure of F is contained in $\Omega(x)$. Then, the segment $[l_0; l[$ is contained in the relative interior of $\Omega(x)$. Now, Lemma 6 shows that $[l_0; l[$ is contained in F. The proposition follows. □

Proposition 9 *The relative interior of a face of a GIT-class is a GIT-class.*

Proof : Let F be a GIT-class. By induction on the codimension of the face in F, it is sufficient to prove the proposition for the maximal faces of F. Let F' be the relative interior of a maximal face of F.

Proposition 8 shows that the closure of F is an union of GIT-classes. But, by Proposition 7, F is the relative interior of its closure. Therefore, the relative boundary of F is an union of GIT-classes. Since, by Proposition 7 the GIT-classes are convex, this implies that the closure of F' is an union of GIT-classes. Moreover, the GIT-classes are open in their closure. So, F' is an union of GIT-classes. Thus, it is sufficient to prove that if l_1 and l_2 belong to F', then they are GIT-equivalent.

Let y be a pivotal point for l_1. Let us prove that l_2 belongs to $\Omega(y)$. If y is semistable for F then l_2 belongs to the closed cone $\Omega(y)$. Otherwise, by Lemma 5 there exists a pivotal point x for F such that $y \in \overline{G \cdot x} - G \cdot x$. Indeed, by Proposition 8, $X^{\rm ss}(F)$ is contained in $X^{\rm ss}(l_1)$. In particular, by Lemma 4, there exists $\lambda \in \mathcal{X}_*(G)$ such that $\Omega(y) = \Omega(x) \cap \{l \in \mathrm{NS}^G(X)_{\mathbb{R}}$ such that $\mu^l(x, \lambda) = 0\}$ and, if l belongs to the relative interior of $\Omega(x)$ then $\mu^l(x, \lambda) < 0$. But, the first assertion of Proposition 6 shows that F is contained in the relative interior of $\Omega(x)$. Thus, the intersection between the closure of F and $\Omega(y)$ is a proper face of F containing l_1. Then, $\overline{F} \cap \Omega(y) = \overline{F'}$. In particular, l_2 belongs to $\Omega(y)$.

We just proved that all pivotal points for l_1 are semistable for l_2. This implies that $X^{\rm ss}(l_1)$ is contained in $X^{\rm ss}(l_2)$. By symmetry, l_1 and l_2 are GIT-equivalent. The proposition is proved. □

5 Global geometry of the GIT-classes

5.1 A notion of wall and chamber

In this section, we introduce a notion of wall and chamber. Definitions close to ours have been considered by I. Dolgachev and Y. Hu in [2] and by M. Thaddeus in [12].

Definition A *wall* is a stability set of codimension one in $C^G(X)$. A *chamber* is a GIT-class of codimension 0 in $C^G(X)$.

By continuity of the functions $\mathrm{M}^\bullet(x)$, the GIT-classes such that $X^{\mathrm{ss}} = X^{\mathrm{s}}$ are chambers. This is the notion of chamber of [2]. But all the chambers are not like that: it is easy to find a G-action such that no GIT-class satisfies $X^{\mathrm{ss}} = X^{\mathrm{s}}$. The appendix of [2] gives an example where some chambers satisfy $X^{\mathrm{ss}} = X^{\mathrm{s}}$ and another does not.

Proposition 10 *The chambers are the connected components of the complement in $C^G(X)$ of the union of the walls. The GIT-classes are the relative interiors of the faces of the chambers.*

Proof : The union of the closures of the chambers is closed. So its complement is an open subset of $C^G(X)$ covered by the GIT-classes of codimension greater that one: this complement is empty. So each GIT-class meets the closure of a chamber. Proposition 9 implies now the second assertion of the proposition. Moreover, each face of codimension one of a chamber is included in a wall. So, the G-ample cone is the disjoint union of the walls and the chambers. The proposition follows immediately. $\square$

Proposition 10 implies that if we can determine the walls, we know all the GIT-classes. This remark is useful to calculate the GIT-classes on examples.

5.2 The GIT-fan

Definition A *fan* Δ in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$ is a finite set of rational convex polyhedral cones in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$ such that

(i) each face of a cone in Δ is also a cone in Δ;

(ii) The intersection of two cones in Δ is a face of each of them.

Most results of this paper about geometry of GIT-classes are summarized in the following theorem announced in the introduction:

Theorem 4 *Let G be a reductive algebraic group acting algebraically on a normal projective variety X. Then:*

(i) For all $l_0 \in C^G(X)$,

$$C(l_0) = \left\{ l \in C^G(X) \text{ such that } X^{\mathrm{ss}}(l_0) \subset X^{\mathrm{ss}}(l) \right\}$$

is a closed convex rational polyhedral cone in $C^G(X)$.

(ii) The cones $C(l)$ form a fan covering $C^G(X)$.

(iii) The GIT-classes are the relative interior of these cones.

This fan is called the GIT-fan for the action of G on X.

Proof : This theorem is a direct consequence of Propositions 7, 8, 9 and 10. □

5.3 Existence of stable points

The following proposition gives an easy criterion on l for the existence of stable points. It was first proved by I. Dolgachev and Y.Hu with slightly different assumptions (see Propositions 3.2.8 and 3.3.5 in [2]).

Proposition 11 *We assume that there exists $l_0 \in C^G(X)$ such that $X^{\mathrm{s}}(l_0)$ is not empty. Then, for $l \in C^G(X)$, $X^{\mathrm{s}}(l)$ is not empty if and only if l belongs to the interior of $C^G(X)$.*

Proof : Let l be a point in $C^G(X)$ such that $X^{\mathrm{s}}(l)$ is not empty. Let $x \in X^{\mathrm{s}}(l)$. By continuity, the function $M^\bullet(x)$ is negative on a neighborhood of l. Hence, l belongs to the interior of $C^G(X)$.

Conversely, let l be a point in the interior of $C^G(X)$. There exists a point l_1 in $C^G(X)$ such that l belongs to the interval $]l_1; l_0]$. Since the sets $X^{\mathrm{ss}}(l_1)$ and $X^{\mathrm{s}}(l_0)$ are open and non empty, there exists $y \in X^{\mathrm{ss}}(l_1) \cap X^{\mathrm{s}}(l_0)$. Then, by convexity of the function $M^\bullet(y)$, y is stable for l. The proposition is proved. □

6 The morphisms induced by inclusions

Propositions 8 and 10 show that for any GIT-class F there exists a chamber C such that $X^{\mathrm{ss}}(C) \subset X^{\mathrm{ss}}(F)$. As a consequence, there exists a morphism $\phi : X^{\mathrm{ss}}(C)//G \longrightarrow X^{\mathrm{ss}}(F)//G$. Moreover, in the proof of Lemma 5, we already observed that ϕ is surjective. So, the quotients corresponding to chambers dominate the other quotients.

It could be interesting to compare two quotients corresponding to two distinct chambers C and C'. In this situation, there exists a sequence of chambers $C = C_0, C_1, \dots, C_m = C'$ such that for all $i = 1, \dots, m$, $C_{i-1} \cap C_i$ is a maximal face of C_{i-1} and C_i; we denote by F_i the relative interior of this face. Then, by Theorem 4, we have: $X^{\mathrm{ss}}(C_0) \subset X^{\mathrm{ss}}(F_1) \supset X^{\mathrm{ss}}(C_1) \subset X^{\mathrm{ss}}(F_2) \supset \cdots \subset X^{\mathrm{ss}}(F_m) \supset X^{\mathrm{ss}}(C_m)$. These inclusions induce a sequence of morphisms:

$$
\begin{tikzcd}
X^{\mathrm{ss}}(C_0)//G \arrow[dr, "\phi_{-,1}"] & & X^{\mathrm{ss}}(C_1)//G \arrow[dl, "\phi_{+,1}"'] \arrow[dr, "\phi_{-,2}"] & \cdots & X^{\mathrm{ss}}(C_m)//G. \arrow[dl, "\phi_{+,m}"'] \\
& X^{\mathrm{ss}}(F_1)//G & & \cdots \; X^{\mathrm{ss}}(F_m)//G &
\end{tikzcd}
\tag{2}
$$

Observe that if there exists $l_0 \in C^G(X)$ such that $X^{\mathrm{s}}(l_0)$ is not empty, then by Proposition 11, $X^{\mathrm{s}}(F_i)$ is not empty, for all $i = 1, \dots, m$. In particular, the morphisms $\phi_{\pm,i}$ are birational. These morphisms are studied in [1], [12], [2] and [13] in various degrees of generality. In this section, our aim is to study the geometry of the fibers of the morphisms $\phi_{\pm,i}$.

More generally, let us fix two effective ample G-linearized line bundles L_1 and L_2 on X such that $X^{\mathrm{ss}}(L_1) \subset X^{\mathrm{ss}}(L_2)$. Consider the following commutative diagram:

$$
\begin{tikzcd}
X^{\mathrm{ss}}(L_1) \arrow[r, hook, "\text{inclusion}"] \arrow[d, "\pi_1"] & X^{\mathrm{ss}}(L_2) \arrow[d, "\pi_2"] \\
X^{\mathrm{ss}}(L_1)//G \arrow[r, "\phi"] & X^{\mathrm{ss}}(L_2)//G
\end{tikzcd}
\tag{3}
$$

We will obtain general descriptions of the fibers of ϕ in terms of quotients of affine subvarieties of X by stabilizers of pivotal points for L_2. Then, we will come back to the situation in (2) and produce pathological examples.

6.1 The fibers of the morphisms induced by inclusions

If H is an algebraic group, we denote by H° the neutral component of H. If V is a H-module, the set of all $v \in V$ such that $0 \in \overline{H.v}$ is called the *nilcone of* V. Let H be a reductive group and Y be a H-variety. If χ denotes a character of H, we denote by L_χ the trivial line bundle on Y linearized by: $h \cdot (y, t) = (h \cdot y, \chi(h)t) \;\; \forall h \in H$, $y \in Y$ and $t \in k$. If Y is affine, we denote by $Y//H$ the affine quotient, namely Spec $\big(k[Y]^H\big)$.

Lemma 7 *Let H be a reductive group and Σ be an affine H-variety (non necessarily irreducible) with a fixed point x. We assume that $\{x\}$ is the only closed orbit of H in Σ. Then:*

(i) The map $\mathcal{X}^*(H) \longrightarrow \mathrm{Pic}^H(\Sigma)$, $\chi \longmapsto L_\chi$ is an isomorphism.

(ii) We have $\Sigma^{\mathrm{ss}}(L_\chi) = \Sigma$ if and only if χ is of finite order.

Proof : Let L be a linearized line bundle on Σ. The action of H on the fiber L_x gives a character χ of H. Since Σ is affine, there exists a section s of L such that $s(x) \neq 0$. The set of zeroes of s is closed, H-stable and does not contain x : it is empty. So L is trivial as a line bundle. This implies the first assertion.

Let $L_\chi \in \mathrm{Pic}^H(\Sigma)$ such that $\Sigma^{\mathrm{ss}}(L_\chi) = \Sigma$. By definition, for some $n > 0$, there exists $s \in \Gamma(\Sigma, L_\chi^{\otimes n})^H$ such that $s(x) \neq 0$. This shows that $n\chi$ is trivial. The second assertion of the lemma follows. □

Proposition 12 *Let L_1, L_2, π_1, π_2 and ϕ like in Diagram 3. Let x be a pivotal point for L_2 which is not semistable for L_1. We denote by H the stabilizer of x. Set $\Sigma = \{y \in X \text{ such that } x \in \overline{H \cdot y}\}$. Then,*

(i) H is reductive,

(ii) Σ is affine,

(iii) $\phi^{-1}(\pi_2(x))$ is isomorphic to $\Sigma^{\mathrm{ss}}(L_{1|\Sigma})//H$.

Moreover, there exists a unique character χ of H of infinite order such that $L_{1|\Sigma} = L_\chi$ in $Pic^H(\Sigma)$. If, in addition X is smooth then the H-variety Σ is isomorphic to the nilcone of a H-module.

Proof : Set $Z = \pi_2^{-1}(\pi_2(x))$. Then Z is affine. Since $G \cdot x$ is closed in Z, it is affine. Hence, Matsushima's theorem shows that H is reductive.

Let $\pi : Z \longrightarrow Z//H$ denote the quotient map. Since $X^{\rm ss}(L_2)$ is open in X and contains x, Σ is contained in $X^{\rm ss}(L_2)$. Then, Σ is contained in Z and $\pi^{-1}(\pi(x)) = \Sigma$. In particular, Σ is affine.

Since Z is affine and contains $G\cdot x$ as a unique closed orbit of G, the Etale Slice Theorem of Luna (see [8]) shows that Z is equivariantly isomorphic to the fiber product $G \times_H \Sigma$. But, by the commutativity of Diagram 3 and the surjectivity of π_1, the fiber $\phi^{-1}(\pi_2(x))$ is equal to $\pi_1(X^{\rm ss}(L_1) \cap Z)$.

Since Z is G-stable and closed in $X^{\rm ss}(L_1)$, $\pi_1(Z \cap X^{\rm ss}(L_1))$ is isomorphic to $Z^{\rm ss}(L_{1|Z})//G$. Moreover, for all $L \in {\rm Pic}^G(G \times_H \Sigma)$ the restriction map from $\Gamma(G \times_H \Sigma, L)^G$ to $\Gamma\left(\Sigma, L_{|\Sigma}\right)^H$ is an isomorphism. We conclude that:

$$Y \simeq Z^{\rm ss}(L_{1|Z})//G \simeq \Sigma^{\rm ss}(L_{1|\Sigma})//H.$$

By Lemma 7, there exists a unique $\chi \in \mathcal{X}^*(H)$ such that $L_{1|\Sigma} = L_\chi$. Since $x \notin \Sigma^{\rm ss}(L_{1|\Sigma})$, Lemma 7 shows that χ is of infinite order.

If in addition X is smooth, the Etale Slice Theorem shows that Σ is equivariantly isomorphic to the nilcone of the normal space $T_xX/T_x(G \cdot x)$ to $G \cdot x$ in X at the point x. The proposition is proved. $\square$

Remark : Conversely, let H be a reductive group, χ be a character of infinite order of H and $\mathcal{N}$ be the nilcone of a H-module V. Then, the variety $\mathcal{N}^{\rm ss}(L_\chi)//H$ occurs as a fiber of a morphism ϕ like in Proposition 12.

Indeed, let $X = \mathbb{P}(V \oplus k)$. We define an action of H on X and a linearization of $\mathcal{O}(1)$ by the formula: $h \cdot (v, t) = (h \cdot v, t)$ for all $h \in H$, $v \in V$ and $t \in k$. We denote by L the line bundle $\mathcal{O}(1)$ so linearized. Let $\pi : X^{\rm ss}(L) \longrightarrow X^{\rm ss}(L)//G$ be the quotient map. It is easy to show that $\pi^{-1}(\pi([0:1])) = \mathcal{N}$. On the other hand, by Proposition 4, for n large enough, $X^{\rm ss}(L^{\otimes n} \otimes \chi)$ is contained in $X^{\rm ss}(L)$. Let ϕ denote the induced morphism from $X^{\rm ss}(L^{\otimes n} \otimes \chi)//G$ to $X^{\rm ss}(L)//G$. Then, by Proposition 12, $\phi^{-1}(\pi([0:1]))$ is isomorphic to $\mathcal{N}^{\rm ss}(L_\chi)//H$.

Proposition 13 *We assume that G is connected. We use the notations and assumptions of Proposition 12. Let Y be an irreducible component of $\phi^{-1}(\pi_2(x))$. Then, there exists an irreducible component S of Σ such that $\pi_1\left(X^{\rm ss}(L_1) \cap \overline{G \cdot S}\right)$ is equal to Y.*

Let H_S denote the stabilizer of S in H. Let χ be as in Proposition 12 and $K_S = {\rm Ker}\chi \cap H_S$. Then, we have:

(i) the group $H_S/K_S \simeq k^*$ acts on $S//K_S$ with a unique closed orbit which is a fixed point.

(ii) $k[S//K_S] = \bigoplus_{n\in\mathbb{N}} k[S//K_S]_n$, where

$$k[S//K_S]_n = \{f \in k[S] \text{ such that } h \cdot f = (\chi(h))^n f \quad \forall h \in H_S\}\,.$$

(iii) $S^{\mathrm{ss}}(L_{1|S})//H_S$ is equal to $\mathrm{Proj}\left(\bigoplus_{n\geq 0} k[S//H_S]_n\right)$.

Moreover, there exists a birational finite morphism from $S^{\mathrm{ss}}(L_{1|S})//H_S$ onto Y.

Proof : Since by Proposition 12, $\phi^{-1}(\pi_2(x)) \simeq \Sigma^{\mathrm{ss}}(L_{1|\Sigma})//H$, the first assertion is obvious. Consider the surjective map:

$$\begin{array}{cccc} \theta\,: & H\times_{H_S} S & \longrightarrow & H\cdot S \\ & (h:s) & \longmapsto & h\cdot s. \end{array}$$

Since H° stabilizes S, H/H_S is finite. In particular, as a variety (without action of H), the fiber product $H \times_{H_S} S$ is isomorphic to $H/H_S \times S$. Thus, θ is finite.

Let $(H \cdot S)_0$ denote the set of all $x \in H \cdot S$ which belong to an unique irreducible component of $H \cdot S$. Set $S_0 = (H \cdot S)_0 \cap S$. One can easily prove that the restriction of θ to $H \times_{H_S} S_0$ is an isomorphism onto $(H \cdot S)_0$. Thus, θ is finite, surjective and birational.

The restriction map $\rho\,:\,\mathrm{Pic}^H(H\times_{H_S} S) \longrightarrow \mathrm{Pic}^{H_S}(S)$ is an isomorphism. We also denote by $L_{1|S}$ the H-linearized line bundle L on $H \times_{H_S} S$ such that $\rho(L) = L_{1|S}$. Then, θ induces a finite birational morphism:

$$\overline{\theta}\,:\,(H\times_{H_S} S)^{\mathrm{ss}}(L_{1|S})//H \longrightarrow (H\cdot S)^{\mathrm{ss}}(L_{1|H\cdot S})//H \simeq Y.$$

Since G has a unique closed orbit in $\pi_2^{-1}(\pi_2(x))$, the point x is the unique closed orbit of H in Σ. As a consequence, $\{x\}$ is the unique closed orbit of H_S in S. Now, $k[S//K_S]^{H_S/K_S} = k[S]^{H_S}$ is equal to k. As a consequence, $S//K_S$ contains a unique closed orbit of H_S/K_S.

Since χ is of infinite order, the group H_S/K_S is isomorphic to k^* via χ. Then, the rational H_S/K_S-module $k[S//K_S]$ is equal to $\bigoplus_{n\in\mathbb{Z}} k[S//K_S]_n$. Let n be a negative integer and $f \in k[S//K_S]_n$. By assumption, $S^{\mathrm{ss}}(L_\chi) = X^{\mathrm{ss}}(L_1) \cap S$ is not empty; so there exists $n_0 > 0$ such that $k[S//K_S]_{n_0}$ contains a non zero function f_0. Then $f^{n_0} f_0^n$ belongs to $k[S//K_S]_0 = k[S]^{H_S}$

and must be constant. By evaluating at x this implies that $f = 0$. Assertion (ii) follows immediately.

Since $\Gamma(S, L_\chi^{\otimes n}) = k[S//K_S]_n$ and $L_{1|S} = L_\chi$, we have:

$$(H \times_{H_S} S)^{\mathrm{ss}}(L_{1|S})//H \simeq S^{\mathrm{ss}}(L_{1|S})//H_S \simeq \mathrm{Proj}\left(\bigoplus_{n\geq 0} k[S//K_S]_n\right).$$

Now, the proposition follows from the properties of $\overline{\theta}$. $\square$

Remark : We will give an example where $\overline{\theta}$ is not an isomorphism in Section 6.4.

6.2 The stabilizers of pivotal points and the stability sets

Proposition 14 *Let $x \in X$. Then there exists a point $y \in X$ such that:*

(i) $x \in \overline{G \cdot y}$

(ii) $\Omega(x) \subset \Omega(y)$

(iii) the interior of $\Omega(y)$ in $C^G(X)$ is not empty

(iv) there exists a character χ of infinite order of G_x such that G_y is contained in the kernel of χ.

Proof : Let us prove the three first assertion by induction on the codimension of $\Omega(x)$. If this codimension is zero, then we can take $y = x$. Otherwise, let l be a point in $\Omega(x)$. Since the codimension of $\Omega(x)$ in $C^G(X)$ is at least one, there exists a line $\mathcal{D}$ such that $\mathcal{D} \cap \Omega(x) = \{l\}$ and $\mathcal{D} \cap C^G(X) \neq \{l\}$. Moreover, by Proposition 4, there exists $l' \in \mathcal{D}$ such that $l' \neq l$ and $X^{\mathrm{ss}}(l') \subset X^{\mathrm{ss}}(l)$. Lemma 5 gives $x' \in X^{\mathrm{ss}}(l')$ such that $x \in \overline{G \cdot x'}$. But now, since $l' \notin \Omega(x)$, $\Omega(x)$ is a proper face of $\Omega(x')$. By induction, the proposition holds for $\Omega(x')$. Therefore, there exists $y \in X$ which satisfies the three first conditions of the proposition.

Let us prove that replacing y by a point of $G \cdot y$, we can obtain the last condition. Let l_1 (resp. l_2) be in the relative interior of $\Omega(y)$ (resp. $\Omega(x)$). Consider the G-variety $\overline{G \cdot y}$ denoted by X_y. Since $G \cdot y$ is dense in X_y,

the variety $X_y^{\rm ss}(l_{2|X_y})//G$ is a point. In particular, $X_y^{\rm ss}(l_{2|X_y})$ is affine and contains $G\cdot x$ as unique closed orbit. Let $\Sigma = \{z \in X_y^{\rm ss}(l_{2|X_y})$ such that $x \in \overline{G_x \cdot z}\}$. By the Etale Slice Theorem, the G-variety $X_y^{\rm ss}(l_{2|X_y})$ is isomorphic to $G\times_{G_x}\Sigma$. Replacing y by a point in $G\cdot y$, we may assume that $y\in\Sigma$. Then, G_y is contained in G_x.

Let L_1 be a G-linearized line bundle on X in the homological class l_1. Let χ be the character of G_x which gives the action of G_x on the fiber L_{1x}. Then, by Proposition 12, the restriction of L_1 to Σ is L_χ and the order of χ is infinite. Moreover, y is semistable for L_1. Thus, there exist a positive integer n and $f\in k[\Sigma]$ such that $h\cdot f = \chi(h)^n f$ for all $h\in G_x$ and $f(y)\neq 0$. In particular, the restriction of χ to G_y is trivial. The proposition is proved. □

Proposition 15 *Let x be a point of X which is pivotal for its stability set. Then the rank of the character group of G_x is at least the codimension of $\Omega(x)$ in the G-ample cone.*

Proof : Let y be a point of X which satisfies the conditions of Proposition 14. Let c denote the codimension of $\Omega(x)$ in $C^G(X)$. There exists a sequence $\Omega(x)=\Omega_0\subset\Omega_1\subset\cdots\subset\Omega_c=\Omega(y)$ of faces of $\Omega(y)$ such that for all $i=1,\dots,c$, the codimension of Ω_{i-1} in Ω_i is equal to one. For all $i=0,\dots,c$, let l_i be a point in the relative interior of Ω_i.

Let H denote the stabilizer of x in G. Consider $X_y=\overline{G\cdot y}$ and $\Sigma_y=\{z\in X_y$ such that $x\in\overline{H\cdot z}\}$. By Proposition 6, for all $i=1,\dots,c$, $X_y^{\rm ss}(l_{i|X_y})$ is strictly contained in $X_y^{\rm ss}(l_{i-1|X_y})$. Then, $X^{\rm ss}(l_0)\cap X_y\simeq G\times_H\Sigma_y$ implies that $\Sigma_y^{\rm ss}(l_{i|\Sigma_y})$ is strictly contained in $\Sigma_y^{\rm ss}(l_{i-1|\Sigma_y})$.

On the other hand, replacing y by a point in $G\cdot y$, we may assume the $y\in\Sigma_y$. Let S_y be an irreducible component of Σ_y containing y. Since $X^{\rm ss}(l_0)\cap X_y$ is isomorphic to the fiber product $G\times_H\Sigma_y$, $G\times_H(H\cdot S_y)$ contains $G\cdot y$. Then, $\Sigma_y=H\cdot S_y$ and thus, for all $i=0,\dots,c$, we have $\Sigma_y^{\rm ss}(l_{i|\Sigma_y})=H\cdot S_y^{\rm ss}(l_{i|S_y})$. In particular, $S_y^{\rm ss}(l_{i|S_y})$ is strictly contained in $S_y^{\rm ss}(l_{i-1|S_y})$, for all $i=1,\dots,c$.

But by Theorem 4, this implies that the $(l_i)_{0\le i\le c}$ are affinely independent in $\mathrm{NS}^{H^\circ}(S_y)_{\mathbb{R}}$. Moreover, by Proposition 12, for all $i=0,\dots,c$, there exists a character χ_i of G_x such that L_{χ_i} belongs to the class $l_{i|\Sigma_y}$. Thus, $\chi_0,\dots,\chi_c$ are affinely independent. The Proposition follows. □

Remark : 1. Proposition 15 is false with $\Omega(x)$ replaced by a GIT-class. See for example: a linear action of the two dimensional torus T on $\mathbb{P}^3$ such that

no weight is contained in the convex hull of the three others.

2. If the codimension of $\Omega(x)$ is equal to the rank of G, Proposition 15 implies that the neutral component of the stabilizer of x is a torus. Arguing as in the proof of the last assertion of Proposition 14, this implies that for all $y \in X$ such that $x \in \overline{G \cdot y}$, the neutral component of G_y is a torus.

6.3 The case where the stabilizer of a pivotal point is a torus

Proposition 16 *With the notations of Propositions 12 and 13, if H° is a torus, then:*

(i) there exists a one-parameter subgroup λ of H such that S is an irreducible component of $\{y \in X : \lim_{t\to 0} \lambda(t)y = x\}$.

(ii) If in addition X is smooth, then S is equivariantly isomorphic to a H_S-module and $S^{\mathrm{ss}}(L_{1|S})//H_S$ is the quotient of a projective toric variety by a finite group. Moreover, this quotient is the normalization of Y.

(iii) If in addition the rank of H° is equal to one, then $\phi^{-1}(\pi_2(x))$ is isomorphic to the quotient of a weighted projective space by a finite group.

Proof : The first assertion is a well-known fact about torus actions. Let us assume that X is smooth. Then, Σ is isomorphic to a nilcone for the action of the torus H°. In particular, the irreducible component S of Σ is equivariantly isomorphic to an H_S-module, denoted by V.

Let $\mathcal{T}$ be a maximal torus of $\mathrm{GL}(V)$ containing the image of H°. Let L_V denote the restriction (after identification of V with S) of L_1 to V. The semistability of a point v in V for L_V depends only on the weights of v for the action of H°. In particular, $\mathcal{T}$ stabilizes the open set $V^{\mathrm{ss}}(L_V)$. Moreover, the actions of H° and of $\mathcal{T}$ on V commute. Hence, $\mathcal{T}$ acts on $V^{\mathrm{ss}}(L_V)//H^\circ$. Because $\mathcal{T}$ has a dense orbit in V, it has a dense orbit in $V^{\mathrm{ss}}(L_V)//H^\circ$, too. Thus, $S^{\mathrm{ss}}(L_{1|S})//H_S$ is the quotient of the projective toric variety $V^{\mathrm{ss}}(L_V)//H^\circ$ by the finite group H_S/H°. Moreover, this quotient is normal. Then, the birational finite morphism of Proposition 13 is the normalization of Y.

From now on, we assume in addition that H° is isomorphic to k^*. Then, S isomorphic to a k^*-module and by Proposition 13, $S^{\mathrm{ss}}(L_{1|S})//H_S$ is the quotient of a weighted projective space by H_S/H°. Moreover, an easy study

of the linear actions of k^* shows that Σ has at most two irreducible components, and that $\Sigma \cap V^{\mathrm{ss}}(L_{1|V})$ is contained in one irreducible component of Σ. In particular, with the notations of Proposition 13, $H_S = H$. But, by Proposition 12, $\phi^{-1}(\pi_2(x))$ is isomorphic to $\Sigma^{\mathrm{ss}}(L_{1|\Sigma})//H$. Then, $\phi^{-1}(\pi_2(x))$ is isomorphic to $S^{\mathrm{ss}}(L_{1|S})//H$. Assertion *(iii)* of the proposition is proved. □

Remark : With additional assumptions, Part *(iii)* of Proposition 16 was proved, by different methods, by M. Thaddeus in [12] and by I. Dolgachev and Y. Hu in [2]. In particular, I. Dolgachev and Y. Hu have shown that the assumptions of Proposition 16 are fulfilled for the diagonal action of G on $X \times G/B$, where B is a Borel subgroup of G.

The following proposition is an application of Propositions 15 and 16 to the actions of SL(2).

Proposition 17 *Let X be a* SL(2)*-variety. Then:*

(i) Any wall is the intersection of an hyperplane and $\mathrm{NS}^G(X)^+_{\mathbb{R}}$.

(ii) Let ϕ be a morphism like in Diagram 3. We assume that X is smooth. Then the fibers of ϕ are weighted projective spaces.

Proof : Since the rank of SL(2) is equal to one, Proposition 15 shows that the codimension of $\Omega(x)$ is less than one for all $x \in X$. Now, by Proposition 6, a wall cannot have a boundary in $\mathrm{NS}^G(X)^+_{\mathbb{R}}$. The first assertion follows immediately.

From now on, we assume that X is smooth. Let L_1, L_2, π_1, π_2, ϕ, x, H and Σ like in Propositions 12. Then, H is a reductive subgroup of SL(2) which has a character of infinite order. This implies easily that H is a maximal torus of SL(2). Then, Propositions 12 and 13 shows that $\phi^{-1}(\pi_2(x))$ is isomorphic to a weighted projective space. □

As an example, we refer to [11] for a detailed study of the diagonal action of SL(2) on $(\mathbb{P}^1)^n$.

6.4 Actions of $k^* \times \mathbf{SL}(2)$

We use the notations of Proposition 12. Let F_1 (resp. F_2) denote the GIT-class of L_1 (resp. L_2). Proposition 12 shows that the fibers of ϕ are simpler

when H is small. Moreover, Propositions 14 and 15 show that H is small when the dimension of $\Omega(x)$ is large. As a consequence, natural restrictions in the study of the morphism ϕ are:

$$\begin{array}{ll} (H_1) & F_1 \text{ is a chamber} \\ (H_2) & \operatorname{codim}(F_2) = 1 \end{array}$$

where $\operatorname{codim}(F_2)$ is the codimension of F_2 in $C^G(X)$. Moreover, these assumptions are fulfilled for the morphisms $\phi_{\pm,i}$ considered in Diagram 2. On the other hand, Propositions 12 and 16 are more precise if X is smooth.

Actually, in [2] and [12] the assumption (H_1) is replaced by: $X^{\mathrm{ss}}(F_1) = X^{\mathrm{s}}(F_1)$. Yet, if one wants to apply Construction 2 to two chambers C and C' such that $X^{\mathrm{ss}}(C) = X^{\mathrm{s}}(C)$ and $X^{\mathrm{ss}}(C') = X^{\mathrm{s}}(C')$, one may have to consider chambers C_i such that $X^{\mathrm{ss}}(C_i) \neq X^{\mathrm{s}}(C_i)$. This happens indeed for $G = k^* \times \mathrm{SL}(2)$, see the example in the appendix of [2].

On the other hand, if G is a torus and (H_1), (H_2) hold, then the fibers of ϕ are weighted projective spaces (this follows from [2] or from Proposition 16). The same holds if $G = \mathrm{SL}(2)$ by Proposition 17. Yet, the following examples show that for the actions of $k^* \times \mathrm{SL}(2)$ on a smooth variety, various varieties can occur as fibers of ϕ, even under the assumptions (H_1) and (H_2). We will give two examples where the morphism ϕ satisfies Assumptions (H_1) and (H_2), whereas ϕ has

(i) a reducible fiber, or

(ii) an irreducible and non normal fiber.

We will need the following technical lemma.

Lemma 8 *Let V be a finite dimensional vector space. Let V_+ and V_- be two vector subspaces. Let H be a reductive group acting on $V_+ \cup V_-$. Let H_+ denote the stabilizer of V_+ in H. We assume that $V_+ \cup V_- = H \cdot V_+$ and that H/H_+ acts trivially on $(V_+ \cap V_-)//H_+$. Then, we have:*

$$(V_+ \cup V_-)//H \simeq V_+//H_+.$$

Proof : We claim that the restriction maps from $k[V_+ \cup V_-]$ to $k[V_+]$ and $k[V_-]$ induces an isomorphism:

$$k[V_+ \cup V_-] \simeq \left\{ (f_+, f_-) \in k[V_+] \times k[V_-] \text{ such that } f_{+|V_+ \cap V_-} = f_{-|V_+ \cap V_-} \right\}.$$

Endeed, let $f_\pm \in k[V_\pm]$ such that $f_{+|V_+\cap V_-} = f_{-|V_+\cap V_-}$. Let W be a vector subspace of V and $W_\pm$ be two vector subspaces of $V_\pm$ such that $V = W \oplus W_+ \oplus W_- \oplus (V_+ \cap V_-)$. We define a function $\tilde{f}$ on V by the formula: $\tilde{f}(w + w_+ + w_- + v) = f_+(w_+ + v) + f_-(w_- + v) - f_+(v)$ for all $w \in W$, $w_\pm \in W_\pm$ and $v \in V_+ \cap V_-$. Then, $\tilde{f}$ is regular on V and the restrictions of $\tilde{f}$ to V_+ and V_- are respectively equal to f_+ and f_-. The claim follows easily.

Now, we consider the morphism $\theta : H \times_{H_+} V_+ \longrightarrow V_+ \cup V_-$ induced by the action of H on $V_+ \cup V_-$. Via the comorphism θ^* of θ, $k[V_+ \cup V_-]$ is identified to a subalgebra of $k[H \times_{H_+} V_+]$. In particular, $\theta^*(k[V_+ \cup V_-]^H) \subset k[H \times_{H_+} V_+]^H \simeq k[V_+]^{H_+}$.

It is sufficient to prove that $\theta^*\left(k[V_+ \cup V_-]^H\right) = k[H \times_{H_+} V_+]^H$. Let $f \in k[H \times_{H_+} V_+]^H$. Let $s \in V_+$ and $h \in H$ such that $h \cdot s \in V_+$. Consider the quotient map, $\pi : V_+ \longrightarrow V_+//H_+$. If $h \in H_+$, then $\pi(s) = \pi(h \cdot s)$. Otherwise, $h \cdot s \in V_+ \cap V_-$. But H/H_{V_+} acts trivially on $(V_+ \cap V_-)//H_+$. Then, $\pi(s) = \pi(h \cdot s)$. Thus, for all $s \in V_+$ and $h \in H$ such that $h \cdot s \in V_+$, we have $f(1 : s) = f(1 : h \cdot s)$. Now, the claim implies that f belongs to $\theta^*\left(k[V_+ \cup V_-]^H\right)$. The lemma follows immediately. $\square$

Examples We begin by fixing some notation. From now on, G denotes the group $k^* \times \mathrm{SL}(2)$. Let χ_0 denote the character of G defined by $\chi_0(t, g) = t$ for all $t \in k^*$ and $g \in \mathrm{SL}(2)$. Let T be the maximal torus of $\mathrm{SL}(2)$ of diagonal matrices and $N(T)$ be its normalizer.

Let $n \in \mathbb{Z}$ and $d \in \mathbb{N}$. We denote by V_d the $\mathrm{SL}(2)$-module of binary forms of degree d in variables a and b. We define an action of k^* on V_d which commutes to the action of $\mathrm{SL}(2)$ by the formula: $t \cdot v = t^n v$ for all $t \in k^*$ and $v \in V_d$. We obtain a G-module denoted by $V_{d,n}$.

Let W be a G-module (two specific choices of W will be given below). Set

$$X = \mathbb{P}(V_{2,0}) \times \mathbb{P}(W \oplus V_{0,1}).$$

Let $\pi_1 : X \longrightarrow \mathbb{P}(V_{2,0})$ and $\pi_2 : X \longrightarrow \mathbb{P}(W \oplus V_{0,1})$ denote the projection maps. Set $L_1 = \pi_1^*(\mathcal{O}(1))$ and $L_2 = \pi_2^*(\mathcal{O}(1))$. We linearize L_1 and L_2 canonically; that is, such that $\Gamma(X, L_1)^*$ is G-isomorphic to $V_{2,0}$ and $\Gamma(X, L_2)^*$ to $W \oplus V_{0,1}$. With the notations of Section 6.1, we have : $\mathrm{Pic}^G(X) = \mathrm{NS}^G(X) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2 \oplus \mathbb{Z}L_{\chi_0}$. Let $\mathbb{R}_{>0}$ denote the interval $]0; +\infty[$. Then,

$$\mathrm{NS}^G(X)^+_{\mathbb{R}} = \mathbb{R}_{>0}L_1 + \mathbb{R}_{>0}L_2 + \mathbb{R}L_{\chi_0}.$$

The decomposition of V_2 in eigenspaces for the action of T is $V_2 = k.a^2 \oplus k.ab \oplus k.b^2$. Denote the elements of X by $([f], [(w, \tau)])$ where $f \in V_2$, $w \in W$

and $\tau \in V_{0,1}$. Consider

$$x_0 = ([ab], [(0,1)]).$$

We have $G_{x_0} = k^* \times N(T)$. We identify, as vector spaces, $T_{x_0}X$ with $(k.a^2 \oplus k.b^2) \times W$, and $T_{x_0}(G \cdot x_0)$ with $k.a^2 \oplus k.b^2$. In particular, $T_{x_0}X/T_{x_0}(G \cdot x_0)$ is isomorphic to W as a vector space. Let $\rho : G \longrightarrow \mathrm{GL}(W)$ denote the morphism induced by the action of G on W. We denote by $W \otimes -\chi_0$ the representation of $k^* \times N(T)$ given by $\tilde{\rho} : k^* \times N(T) \longrightarrow \mathrm{GL}(W)$, $(t_0, t_1) \mapsto t_0^{-1}\rho((t_0, t_1))$. Then, we have the following isomorphism of $k^* \times N(T)$-modules:

$$T_{x_0}X/T_{x_0}(G \cdot x_0) \simeq W \otimes -\chi_0.$$

Let m and n be positive integers. Let $V^{m,n}$ denote the G-module $V_{2,0}^{\otimes m} \otimes (W \oplus V_{0,1})^{\otimes n}$. Consider the Segre embedding i of X in $\mathbb{P}(V^{m,n})$. Let $L_{m,n}$ be the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(V^{m,n})$ canonically linearized; in particular, $i^*(L_{m,n}) = mL_1 + nL_2$ in $\mathrm{Pic}^G(X)$.

Let us use the notations of Section 1.3 for the action of $k^* \times T$ on $V^{m,n}$. For all $x \in X$, the vertices of $\mathrm{Conv}(\mathrm{st}(i_{m,n}(x)))$ belong to the set:

$$\mathcal{V} = \{m\alpha + n\beta \text{ such that } \alpha \in \mathrm{st}(V_{2,0}) \text{ and } \beta \in \mathrm{st}(W \oplus V_{0,1})\}.$$

Let χ_1 denote the character of $k^* \times T$ defined by: $\chi_1(t_0, t_1) = t_1$ for all $(t_0, t_1) \in k^* \times T$. Then, $\mathcal{X}^*(k^* \times T) = \mathbb{Z}\chi_0 \oplus \mathbb{Z}\chi_1$. We have: $\mathrm{st}(i_{m,n}(x_0)) = \{n\chi_0\}$. Moreover, for all $g \in G$, $n\chi_0$ belongs to $\mathrm{st}(i_{m,n}(g \cdot x_0))$. One can easily conclude that a point $L_1 + qL_2 + \theta L_{\chi_0} \in \mathrm{NS}^G(X)^+_{\mathbb{R}}$ belongs to $\Omega(x_0)$ if and only if $q = \theta$.

A first choice of W

Set

$$W = V_{1,-3} \oplus V_{1,-1} \oplus V_{1,1} \oplus V_{1,3}.$$

The weights of the action of $k^* \times T$ on $W \otimes -\chi_0$ are the crosses on Figure 1. The meaning of the polytopes $\mathcal{N}_{1,u}$, $\mathcal{N}_{2,u}$ and $\mathcal{N}_{3,d}$ on Figure 1 will be explained later.

Set $q = n/m$. The set $\mathcal{V}$ is represented by crosses on Figure 2 after dilatation of ratio $1/m$.

Set $l_0 = L_1 + 36(L_2 + L_{\chi_0})$. Note that l_0 belongs to $\Omega(x_0)$. Set $l_- = l_0 - L_{\chi_0}$ and $l_+ = l_0 + L_{\chi_0}$. We denote respectively by $C_\pm$ the GIT-class of $l_\pm$. One can easily see on Figure 2 that for $q = 36$ the points $35\chi_0$ and $37\chi_0$ do not belong to the boundary of a polytope with vertices in $\mathcal{V}$. We conclude

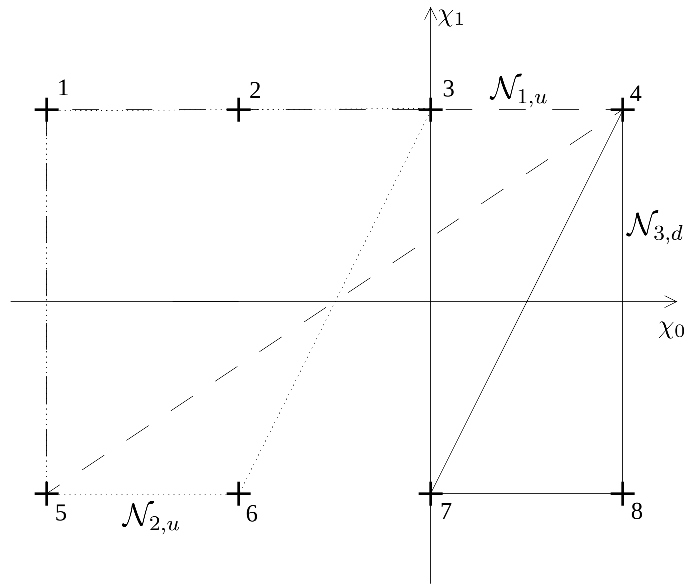

Figure 1: Weights of $k^* \times T$ in $W \otimes -\chi_0$.

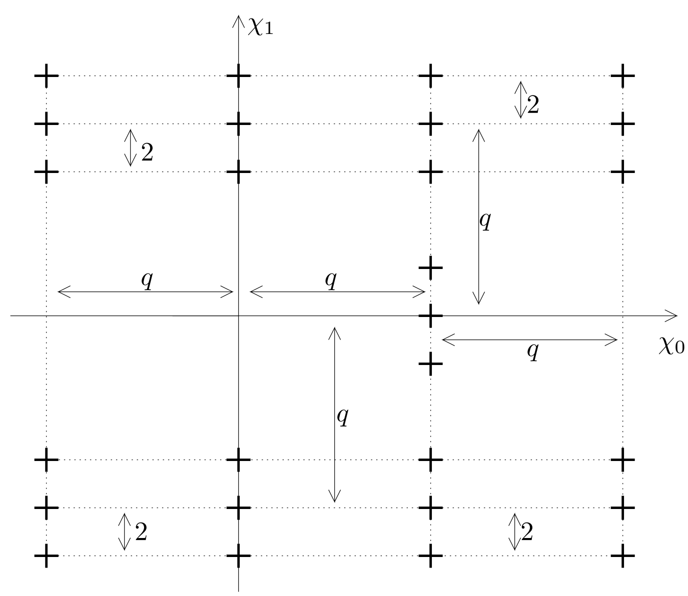

Figure 2: The set $\mathcal{V}$.

that C_+ and C_- are chambers such that $X^{\mathrm{ss}}(C_\pm) = X^{\mathrm{s}}(C_\pm)$. By similar arguments one can show that the GIT-class F_0 of l_0 is a face of C_+ and C_-. Moreover, the codimension of F_0 in $C^G(X)$ is equal to one. In particular, we have: $X^{\mathrm{ss}}(C_-) \subset X^{\mathrm{ss}}(F_0) \supset X^{\mathrm{ss}}(C_+)$. These inclusions induce a diagram:

$$\begin{tikzcd} X^{\mathrm{ss}}(C_-)//G \arrow[dr, "\phi_-"'] & & X^{\mathrm{ss}}(C_+)//G \arrow[dl, "\phi_+"] \\ & X^{\mathrm{ss}}(F_0)//G & \end{tikzcd}$$

Let $\mathcal{N}$ denote the nilcone of $W \otimes -\chi_0$ for the action of $k^* \times N(T)$. Let $(\epsilon_1, \dots, \epsilon_8)$ be a base of W of eigenvectors for $k^* \times T$ such that the weight of ϵ_i is the cross i on Figure 1. Let $(x_1, \dots, x_8)$ denote the dual basis of $(\epsilon_1, \dots, \epsilon_8)$. A vector $v \in W$ belongs to $\mathcal{N}$ if and only if 0 does not belong to the convex hull of the weights of v for $k^* \times T$. In particular, the irreducible components of $\mathcal{N}$ correspond to the maximal convex hulls of weights of $k^* \times T$ which do not contain 0. We obtain six irreducible components for $\mathcal{N}$. The equations of these six components are:

$$\begin{array}{llll} \mathcal{N}_{1,u} & : x_6 = x_7 = x_8 = 0 & \mathcal{N}_{1,d} & : x_2 = x_3 = x_4 = 0 \\ \mathcal{N}_{2,u} & : x_4 = x_7 = x_8 = 0 & \mathcal{N}_{2,d} & : x_3 = x_4 = x_8 = 0 \\ \mathcal{N}_{3,u} & : x_1 = x_2 = x_5 = x_6 = x_7 = 0 & \mathcal{N}_{3,d} & : x_1 = x_2 = x_3 = x_5 = x_6 = 0 \end{array}$$

The convex hull of the weights of the action of $k^* \times T$ on $\mathcal{N}_{1,u}$, $\mathcal{N}_{2,u}$ and $\mathcal{N}_{3,d}$ are represented on Figure 1.

Proposition 12 shows that:

$$\phi_\pm^{-1}(\pi_0(x_0)) \simeq \mathcal{N}^{\mathrm{ss}}\left(L_{\pm\chi_0}\right)//(k^* \times N(T)). \tag{4}$$

In particular,

$$\phi_+^{-1}(\pi_0(x_0)) \simeq (\mathcal{N}_{3,u} \cup \mathcal{N}_{3,d})^{\mathrm{ss}}\left(L_{\chi_0}\right)//(k^* \times N(T)).$$

Moreover, by Lemma 8, we have $(\mathcal{N}_{3,u} \cup \mathcal{N}_{3,d})//N(T) \simeq \mathcal{N}_{3,u}//T$. This implies that $\phi_+^{-1}(\pi_0(x_0)) \simeq \mathcal{N}_{3,u}^{\mathrm{ss}}(L_{\chi_0})//(k^* \times T)$. In particular, this fiber is a projective toric variety of dimension one. Then, we have

$$\phi_+^{-1}(\pi_0(x_0)) \simeq \mathbb{P}^1.$$

By Isomorphism 4, $(\mathcal{N}_{1,u}\cup\mathcal{N}_{1,d})^{\rm ss}(L_{-\chi_0})//(k^*\times N(T))$ is isomorphic to an irreducible component of $\phi_-^{-1}(\pi_0(x_0))$. Write $k[\mathcal{N}_{1,u}]=k[x_1,\dots,x_5]$. The vector space $k[\mathcal{N}_{1,u}]^T$ is generated by the monomials $x_1^{n_1}\cdots x_5^{n_5}$ such that $n_1+\cdots+n_4=n_5$. Moreover, the weight of such a monomial for $k^*\times T$ is $-2(4n_1+3n+2n_3+n_4)\chi_0$. In particular, the quotient $\mathcal{N}_{1,u}^{\rm ss}(L_{-\chi_0})//(k^*\times T)$ is isomorphic to the weighted projective space $\mathbb{P}(1,2,3,4)$ and $N(T)/T$ acts trivially on $(\mathcal{N}_{1,u}\cap\mathcal{N}_{1,d})//T$. Then, Lemma 8 allows us to conclude that this irreducible component of $\phi_-^{-1}(\pi_0(x_0))$ is isomorphic to $\mathbb{P}(1,2,3,4)$.

Let $\mathbb{R}_{\geq 0}$ denote the set of non negative real numbers. If d is a non negative integer, we denote by $\mathcal{P}_d$ the set of $(n_1,n_2,n_3,n_5,n_6)\in\mathbb{R}^5_{\geq 0}$ such that:

$$\begin{array}{rl} & n_1+n_2+n_3=n_5+n_6\\ \text{and} & 4(n_1+n_5)+2n_2+2n_6=d.\end{array}$$

Then, $\mathcal{P}_d$ is a convex polytope.

One can easily prove that $\mathcal{N}_{2,u}^{\rm ss}(L_{-\chi_0})//(k^*\times T)$ is isomorphic to

$$\mathrm{Proj}\left(\bigoplus_{d\geq 0}\ \bigoplus_{(n_1,n_2,n_3,n_5,n_6)\in\mathcal{P}_d\cap\mathbb{Z}^5} k.x_1^{n_1}x_2^{n_2}x_3^{n_3}x_5^{n_5}x_6^{n_6}\right).$$

Since the polytopes $\mathcal{P}_d$ are not simplicial, this toric variety is not a weighted projective space. Moreover, Isomorphism 4 and Lemma 8 imply that an irreducible component of $\phi_-^{-1}(\pi_0(x_0))$ is isomorphic to this toric variety.

A second choice of W

Set

$$W=V_{1,-1}\oplus V_{1,1}\oplus V_{3,3}.$$

The weights of the action of $k^*\times T$ on $W\otimes -\chi_0$ are the crosses on Figure 3.

Like in the previous example, we set $l_0=L_1+36(L_2+L_{\chi_0})$, $l_\pm=l_0\pm L_{\chi_0}$. The GIT-classes of $l_\pm$ are two chambers $C_\pm$ and the GIT-class of l_0 is a maximal face F_0 of C_+ and C_-, too. Let $\phi_+ : X^{\rm ss}(C_+)//G\longrightarrow X^{\rm ss}(F_0)//G$ denote the morphism induced by the inclusion $X^{\rm ss}(C_+)\subset X^{\rm ss}(F_0)$.

The nilcone $\mathcal{N}$ of $W\otimes -\chi_0$ has four irreducible components. With obvious notations, the equations of these components are:

$$\begin{array}{llll}\mathcal{N}_{1,u} : & x_1=x_2=x_3=x_4=x_5=0 & \mathcal{N}_{1,d} : & x_1=x_2=x_3=x_4=x_8=0\\ \mathcal{N}_{2,u} : & x_5=x_6=x_7=0 & \mathcal{N}_{2,d} : & x_6=x_7=x_8=0\end{array}$$

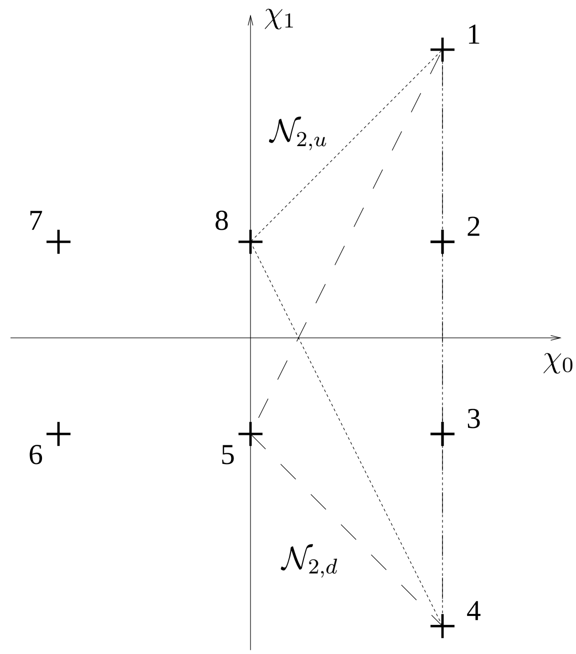

Figure 3: Weights of $k^* \times T$ in $W \otimes -\chi_0$.

By Proposition 12, we have:

$$\phi_+^{-1}(\pi_0(x_0)) \simeq (\mathcal{N}_{2,u} \cup \mathcal{N}_{2,d})^{\mathrm{ss}} (L_{\chi_0})//(k^* \times N(T)).$$

Moreover, by Proposition 13, the natural map

$$\overline{\theta} : \mathcal{N}_{2,u}^{\mathrm{ss}}(L_{\chi_0})//(k^* \times T) \longrightarrow (\mathcal{N}_{2,u} \cup \mathcal{N}_{2,d})^{\mathrm{ss}} (L_{\chi_0})//(k^* \times N(T))$$

is birational and finite.

The restriction of $\overline{\theta}$ to $(\mathcal{N}_{2,u} \cap \mathcal{N}_{2,d})^{\mathrm{ss}} (L_{\chi_0})//(k^* \times T)$ is the quotient by the action of $N(T)/T$. But we have:

$$k[\mathcal{N}_{2,u} \cap \mathcal{N}_{2,d}]^T = k[x_1x_4,\, x_2x_3,\, x_1x_3^3,\, x_4x_2^3].$$

Let $\alpha : k[X_1,\, X_2,\, X_3,\, X_4] \longrightarrow k[x_1x_4,\, x_2x_3,\, x_1x_3^3,\, x_4x_2^3]$ be the morphism defined by $\alpha(X_1) = x_1x_4$, $\alpha(X_2) = x_2x_3$, $\alpha(X_3) = x_1x_3^3$ and $\alpha(X_4) = x_4x_2^3$. Then, α induces an isomorphism $\bar{\alpha}$ from $k[X_1,\, X_2,\, X_3,\, X_4]/(X_3X_4 - X_1X_2^3)$ onto $k[\mathcal{N}_{2,u} \cap \mathcal{N}_{2,d}]^T$.

Moreover,

$$k[\mathcal{N}_{2,u} \cap \mathcal{N}_{2,d}]^{N(T)} = \bar{\alpha}\left(k[X_1,\, X_2,\, X_3 + X_4]\right),$$

which is strictly contained in $k[\mathcal{N}_{2,u} \cap \mathcal{N}_{2,d}]^T$. In particular, $\overline{\theta}$ is not an isomorphism. Since, $\mathcal{N}_{2,u}^{\mathrm{ss}}(L_{\chi_0})//(k^* \times T)$ is normal, this implies that $\phi_+^{-1}(\pi_0(x_0))$ is irreducible but not normal.

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